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The Structure of Translation-Invariant Memories: Two Addenda

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1. INTRODUCTION

This note is a supplement to an earlier paper [1], in which Coffman and Schäffer examined the structure of the "memory" operator that occurs in an autonomous linear retarded functional–differential equation in a Banach space under Carathéodory conditions, when the recall of the "memory" is bounded (say by 1). It is easy to see that the structure of such a "memory" can be completely accounted for by examining its action on periodic functions with a suitable period (specifically, 2); this action is a so-called P-memory [1, p. 435 and Lemmas 8.1, 8.2].

One question that was raised in [1] was whether the assumptions entering the definition of a P-memory necessarily entailed its boundedness as an operator from the space of continuous to that of integrable periodic functions with values in the given Banach space. It was added in proof [1, p. 453] that the answer is affirmative; this result was communicated by B. E. Johnson, who indicated its derivation from a theorem of his, unpublished at the time, concerning translation-invariant mappings between spaces of periodic functions. Our first addendum (Sections 2–4) is devoted to an account of this theorem (Theorem 3.2) and its consequences for P-memories, as well as of some related corollaries. This first addendum uses only Sections 2–4 of [1].

The second addendum (Section 5) consists of an example of a P-memory in the scalar case; it complements [1, Examples 7.8 and 7.9]. In the earlier instances, the P-memory did not map all continuous functions into continuous

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functions; in the present example, it does not even map all continuous functions into qth-power-integrable functions for any $q > 2$, and lacks all extensions to bounded linear mappings between Lebesgue spaces of periodic functions other than those possessed, on general principles, by every bounded linear translation-invariant mapping [1, Lemmas 3.2, 3.5, 3.7]. Our present example, however, is not quite so explicit in its structure as the previous ones. In its discussion we shall require a representation theorem for P-memories: specifically, [1, Corollary 7.2].

2. Preliminaries and Two Lemmas

The terminology and notation in [1, Sections 2–4] are used, with the following additions. C denotes the field of complex numbers, also considered as a complex Banach space with the norm $|\ |$. $l^1(\mathbb{Z})$ denotes the space of absolutely summable functions $\xi: \mathbb{Z} \to C$ with the norm $|\xi|_1 = \sum_{n \in \mathbb{Z}} |\xi(n)|$. If X is a Banach space, $1_X \in [X \to X]$ is the identity mapping on X. A set $G \subset R$ is *periodic* if $2 + G = G$ (i.e., if χ_G has period 2).

For the given Banach space E and for each $n \in \mathbb{Z}$, we define the operator $\mathscr{F}_n \in [\mathbf{P}^1(E) \to E]$ by

$$\mathscr{F}_n f = \tfrac{1}{2} \int (Z_{-n} 1) \cdot f \qquad \left(= \tfrac{1}{2} \int e^{-\pi i n s} f(s)\, ds\right), \qquad f \in \mathbf{P}^1(E), \quad n \in \mathbb{Z},$$

the nth Fourier coefficient of f; we note that $\| \mathscr{F}_n \| = \frac{1}{2}$ for each n, and record the formulas

$$\mathscr{F}_n T_t = e^{-\pi i n t} \mathscr{F}_n, \qquad \mathscr{F}_n Z_m = \delta_{mn} 1_E \qquad \text{for } m, n \in \mathbb{Z},\ t \in R. \tag{2.1}$$

We also record a standard "uniqueness" result.

$$\text{If } f \in \mathbf{P}^1(E) \text{ and } \mathscr{F}_n f = 0 \text{ for all } n \in \mathbb{Z}, \text{ then } f = 0. \tag{2.2}$$

The proof of this result can be obtained by some of the methods used for proving it when $E = C$ (e.g., [3, Section 5.1.2]); or it may be reduced to that special case by composition with arbitrary bounded linear functionals on E.

For each set $K \subset \mathbb{Z}$ we consider $\mathbf{Y}_K = \{f \in \mathbf{P}^1(E): \mathscr{F}_n f = 0 \text{ for all } n \in K\}$, a closed subspace of $\mathbf{P}^1(E)$; and the quotient mapping $\Omega_K \in [\mathbf{P}^1(E) \to \mathbf{P}^1(E)/\mathbf{Y}_K]$. If X is a Banach space and $F: X \to \mathbf{P}^1(E)$ is a linear mapping, not necessarily bounded, set $A_F = \{n \in \mathbb{Z}: \mathscr{F}_n F: X \to E \text{ is unbounded}\}$.

We next prove two lemmas, which are specialized versions of [4, Lemma 7.3, Theorem 7.1].

LEMMA 2.1. *Let X be a Banach space and $F\colon X \to \mathbf{P}^1(E)$ a linear mapping. If $K \subset \mathbb{Z}$, then $\Omega_K F \in [X \to \mathbf{P}^1(E)/\mathbf{Y}_K]$ if and only if $K \cap A_F = \varnothing$.*

Proof. Assume that $\Omega_K F \in [X \to \mathbf{P}^1(E)/\mathbf{Y}_K]$ and let $n \in K$ be given. The kernel of $\mathscr{F}_n$ is $\mathbf{Y}_{\{n\}}$, which includes $\mathbf{Y}_K$; there exists, therefore, an operator $U \in [\mathbf{P}^1(E)/\mathbf{Y}_K \to E]$ such that $\mathscr{F}_n = U\Omega_K$. But then $\mathscr{F}_n F = U(\Omega_K F) \in [X \to E]$, so that $n \notin A_F$. Since $n \in K$ was arbitrary, we conclude that $K \cap A_F = \varnothing$.

Assume conversely that $K \cap A_F = \varnothing$; it will be enough to show that the graph of $\Omega_K F\colon X \to \mathbf{P}^1(E)/\mathbf{Y}_K$ is closed. Let (x_k), a sequence in X, and $g \in \mathbf{P}^1(E)$ be such that $\lim_{k\to\infty} x_k = 0$ in X and $\lim_{k\to\infty} \Omega_K F x_k = \Omega_K g$ in $\mathbf{P}^1(E)/\mathbf{Y}_K$. It remains to show that $\Omega_K g = 0$, i.e., that $g \in \mathbf{Y}_K$.

Let $n \in K$ be given, and let U be as in the first part of the proof. Now $n \in K \subset \mathbb{Z}\backslash A_F$, so that $\mathscr{F}_n F \in [X \to E]$; hence,

$$\mathscr{F}_n g = U\Omega_K g = \lim_{k\to\infty} U(\Omega_K F x_k) = \lim_{k\to\infty} (\mathscr{F}_n F)\, x_k = 0 \text{ in } E;$$

since this holds for all $n \in K$, we have $g \in \mathbf{Y}_K$, as was to be proved.

LEMMA 2.2. *Let $a \in R$ be given, and let $\mathbf{X} \subset \mathbf{P}^1(E)$ be a set satisfying*

$$\sum_{n\in\mathbb{Z}} \xi(n)\, T_{na} f \in \mathbf{X} \qquad \textit{for each } f \in \mathbf{X} \textit{ and } \xi \in l^1(\mathbb{Z}); \tag{2.3}$$

let $G \subset R$ be an open periodic set, and put $K = \{n \in \mathbb{Z}\colon na \in G\}$. Then

$$\bigcap \{(T_a - e^{-\pi i s})(\mathbf{X})\colon s \in G\} = \mathbf{X} \cap \mathbf{Y}_K. \tag{2.4}$$

Proof. If $g \in \mathbf{X}$, then $(T_a - e^{-\pi i s})\, g \in \mathbf{X}$ for all $s \in R$, by (2.3); and, using (2.1), we infer that $\mathscr{F}_n(T_a - e^{-\pi i n a})\, g = 0$ for each $n \in \mathbb{Z}$. Therefore the intersection on the left in (2.4) is included in $\mathbf{X} \cap \mathbf{Y}_K$.

To prove the reverse inclusion, assume that $f \in \mathbf{X} \cap \mathbf{Y}_K$ and $s \in G$ are given. Since G is open and periodic, there exists a function $\varphi \in \mathbf{PC}(C)$ that is continuously differentiable (or even smoother, if we insist) and that satisfies

$$(e^{-\pi i t} - e^{-\pi i s})\, \varphi(t) = 1 \qquad \text{for all } t \in R\backslash G. \tag{2.5}$$

As is well known from Fourier Analysis, the smoothness of φ implies

$$\varphi = \sum_{n\in\mathbb{Z}} Z_n \xi(n) \qquad \text{for some } \xi \in l^1(\mathbb{Z}). \tag{2.6}$$

Set $g = \sum_{n\in\mathbb{Z}} \xi(n)\, T_{na}f$; by (2.3) we have $g \in \mathbf{X}$. Using (2.1), (2.6) and the boundedness of each $\mathscr{F}_n$, we find

$$\begin{aligned}\mathscr{F}_n(T_a - e^{-\pi i s})g &= (e^{-\pi i n a} - e^{-\pi i s}) \sum_{m\in\mathbb{Z}} \xi(m)\, \mathscr{F}_n T_{ma} f \\ &= (e^{-\pi i n a} - e^{-\pi i s}) \sum_{m\in\mathbb{Z}} e^{-\pi i n m a} \xi(m)\, \mathscr{F}_n f \\ &= (e^{-\pi i n a} - e^{-\pi i s})\, \varphi(na)\, \mathscr{F}_n f = \mathscr{F}_n f \qquad \text{for each } n \in \mathbb{Z};\end{aligned}$$

the last equality follows from (2.5) when $na \in R\backslash G$ and from $f \in \mathbf{Y}_K$ when $na \in G$. By (2.2) it follows that $f = (T_a - e^{-\pi i s})\, g \in (T_a - e^{-\pi i s})(\mathbf{X})$. Since $s \in G$ and $f \in \mathbf{X} \cap \mathbf{Y}_K$ were arbitrary, the desired reverse inclusion between the members of (2.4) holds.

3. Translation-Invariant Operators

Let us consider the following objects:

(i) an irrational number $a \in R$;

(ii) a linear manifold $\mathbf{X} \subset \mathbf{P}^1(E)$ provided with a norm $|\ |_{\mathbf{X}}$ making it into a Banach space, and such that for every $f \in \mathbf{X}$ we have $T_a f,\ T_{-a} f \in \mathbf{X}$ with $|T_a f|_{\mathbf{X}} = |f|_{\mathbf{X}}$; thus, $|T_{na} f|_{\mathbf{X}} = |f|_{\mathbf{X}}$ for all $n \in \mathbb{Z}$, and $\mathbf{X}$ satisfies (2.3);

(iii) a linear mapping $F\colon \mathbf{X} \to \mathbf{P}^1(E)$, not necessarily bounded, satisfying

$$T_a F = F T_a. \tag{3.1}$$

Lemma 3.1. *If a, $\mathbf{X}$, F are as in* (i)–(iii), *then A_F is finite.*

Proof. (1) Suppose A_F is infinite. Since a is irrational, the mapping $(k, n) \mapsto ka + 2n\colon \mathbb{Z} \times \mathbb{Z} \to R$ is injective. It follows that the set $\{ka + 2n\colon k \in A_F,\ n \in \mathbb{Z}\} \cap [-1, 1]$ is infinite, and we may choose an (infinite) set $L \subset A_F$ such that $\{ka + 2n\colon k \in L,\ n \in \mathbb{Z}\} \cap [-1, 1]$ is infinite and discrete. We infer from all this that we can find open periodic sets $G_k \subset R$ for all $k \in L$, such that

$$ka \in G_k \quad \text{and} \quad \operatorname{cl} G_k \cap \operatorname{cl} \bigcup \{G_j\colon j \in L\backslash\{k\}\} = \varnothing \qquad \text{for each } k \in L. \tag{3.2}$$

We may then choose functions $\varphi_k \in \mathbf{PC}$, $k \in L$, that are continuously differentiable (or even smoother, if we please) and such that, for each $k \in L$,

$$\varphi_k(t) = \delta_{jk} \qquad \text{for all } t \in G_j,\ j \in L. \tag{3.3}$$

(The construction of these functions is simplified if we choose each set G_k, as we may, to meet $[ka - 1, ka + 1]$ in a single open interval.) As in the proof of Lemma 2.2, Fourier Analysis shows that, on account of the smoothness of φ_k,

$$\varphi_k = \sum_{n\in\mathbb{Z}} Z_n \xi_k(n) \qquad \text{for some } \xi_k \in l^1(\mathbb{Z}) \qquad \text{for each } k \in L. \tag{3.4}$$

(2) For each $k \in L$ set $K_k = \{n \in \mathbb{Z}: na \in G_k\}$, $\mathbf{Y}_k = \mathbf{Y}_{K_k}$, and $\Omega_k = \Omega_{K_k}$; we note that the family $(K_k : k \in L)$ is pairwise disjoint, by (3.2). Since $ka \in G_k$ we have $k \in L \cap K_k \subset A_F \cap K_k$. By Lemma 2.1, we infer that $\Omega_k F: \mathbf{X} \to \mathbf{P}^1(E)/\mathbf{Y}_K$ is unbounded. Since L is infinite, we may therefore choose $g_k \in \mathbf{X}$ for each $k \in L$ so that simultaneously

$$\sum_{k\in L} | \xi_k |_1 \, | g_k |_{\mathbf{X}} < \infty, \tag{3.5}$$

$$\text{the set } \{\| \Omega_k F g_k \|_{\mathbf{P}^1(E)/\mathbf{Y}_k} : k \in L\} \text{ is unbounded.} \tag{3.6}$$

We set $g = \sum_{k\in L} \sum_{n\in\mathbb{Z}} \xi_k(n)\, T_{na} g_k$; by the assumptions on $\mathbf{X}$ and by (3.5), $g \in \mathbf{X}$. For each $k \in L$ and each $n \in K_k$ we find, using the boundedness of each $\mathscr{F}_n$ and (2.1), (3.4), (3.3),

$$\mathscr{F}_n g = \sum_{j\in L} \sum_{m\in\mathbb{Z}} \xi_j(m)\, e^{-\pi i n m a} \mathscr{F}_n g_j = \sum_{j\in L} \varphi_j(na)\, \mathscr{F}_n g_j = \mathscr{F}_n g_k;$$

by Lemma 2.2 this implies

$$g - g_k \in \mathbf{X} \cap \mathbf{Y}_k = \bigcap \{(T_a - e^{-\pi i s})(\mathbf{X}): s \in G_k\} \qquad \text{for each } k \in L. \tag{3.7}$$

On the other hand, (3.1) and Lemma 2.2 with $\mathbf{P}^1(E)$ itself in the role of $\mathbf{X}$ yield

$$F\left(\bigcap \{(T_a - e^{-\pi i s})(\mathbf{X}): s \in G_k\}\right) \subset \bigcap \{(T_a - e^{-\pi i s}) F(\mathbf{X}): s \in G_k\}$$

$$\subset \bigcap \{(T_a - e^{-\pi i s})(\mathbf{P}^1(E)): s \in G_k\} = \mathbf{Y}_k \qquad \text{for each } k \in L. \tag{3.8}$$

Combining (3.7) and (3.8), we find $F(g - g_k) \in \mathbf{Y}_k$, and therefore, $\Omega_k F g = \Omega_k F g_k$ for each $k \in L$. But then

$$\| \Omega_k F g_k \|_{\mathbf{P}^1(E)/\mathbf{Y}_k} = \| \Omega_k F g \|_{\mathbf{P}^1(E)/\mathbf{Y}_k} \leqslant | Fg |_1 \qquad \text{for all } k \in L,$$

and this contradicts (3.6).

THEOREM 3.2. *Let* a, $\mathbf{X}$, F *be as in* (i)–(iii). *Then the set* A_F *is finite, and* $F - \sum_{n\in A_F} Z_n \mathscr{F}_n F \in [\mathbf{X} \to \mathbf{P}^1(E)]$.

Proof. A_F is finite by Lemma 3.1. The operator $1_{\mathbf{P}^1(E)} - \sum_{n\in A_F} Z_n\mathscr{F}_n \in [\mathbf{P}^1(E) \to \mathbf{P}^1(E)]$ (a projection) has the kernel $\mathbf{Y}_{\mathbb{Z}\setminus A_F}$, on account of (2.1) and (2.2). There exists, therefore, an operator $J \in [\mathbf{P}^1(E)/\mathbf{Y}_{\mathbb{Z}\setminus A_F} \to \mathbf{P}^1(E)]$ such that $1_{\mathbf{P}^1(E)} - \sum_{n\in A_F} Z_n\mathscr{F}_n = J\Omega_{\mathbb{Z}\setminus A_F}$. But then Lemma 2.1 implies

$$F - \sum_{n\in A_F} Z_n\mathscr{F}_n F = (J\Omega_{\mathbb{Z}\setminus A_F})F = J(\Omega_{\mathbb{Z}\setminus A_F}F) \in [\mathbf{X} \to \mathbf{P}^1(E)].$$

COROLLARY 3.3. *Let* $\mathbf{X}$ *be one of the spaces* $\mathbf{P}^p(E)$, $1 \leqslant p \leqslant \infty$, *or* $\mathbf{PC}(E)$, *and let* $F\colon \mathbf{X} \to \mathbf{P}^1(E)$ *be linear and translation-invariant. Then* A_F *is finite, and* $F - \sum_{n\in A_F} Z_n\mathscr{F}_n F \in [\mathbf{X} \to \mathbf{P}^1(E)]$ *is bounded and translation-invariant.*

Proof. $\mathbf{X}$ and F obviously satisfy assumptions (ii), (iii) for every irrational $a \in R$, so that Theorem 3.2 is applicable. It remains to show that $F - \sum_{n\in A_F} Z_n\mathscr{F}_n F$ is translation-invariant. Now the translation-invariance of F, together with (2.1) and [1, (2.7)], imply

$$Z_n\mathscr{F}_n FT_t = Z_n\mathscr{F}_n T_t F = e^{-\pi int} Z_n\mathscr{F}_n F = T_t Z_n\mathscr{F}_n F \qquad \text{for all } t \in R,\ n \in \mathbb{Z};$$

thus each $Z_n\mathscr{F}_n F$ is translation-invariant, and the conclusion follows.

We conclude this section with a theorem that considerably strengthens [1, Lemma 3.2]; for the definition of F_{pq}, see that lemma.

THEOREM 3.4. *Suppose* $F \in [\mathbf{PC}(E) \to \mathbf{P}^1(E)]$ *is translation-invariant; suppose that* p, q *are given, with* $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, *and that there exists a linear translation-invariant mapping* $F'\colon \mathbf{P}^p(E) \to \mathbf{P}^q(E)$ *such that* $F'f = Ff$ *for all* $f \in \mathbf{PC}(E)$. *Then* F_{pq} *exists.*

Proof. Let $F''\colon \mathbf{P}^p(E) \to \mathbf{P}^1(E)$ be the composition of F' with the inclusion $\mathbf{P}^q(E) \to \mathbf{P}^1(E)$. F'' is translation-invariant, and Corollary 3.3 is applicable to it. Write A for the finite set $A_{F''}$; thus $F'' - \sum_{n\in A} Z_n\mathscr{F}_n F'' \in [\mathbf{P}^p(E) \to \mathbf{P}^1(E)]$ is translation-invariant.

By [1, Lemma 3.1], there exists for each $n \in \mathbb{Z}$ an operator $Q_n \in [E \to E]$ such that $F''Z_n = FZ_n = Z_nQ_n$. We set

$$\begin{aligned} F^0 &= F'' + \sum_{n\in A} Z_n(Q_n\mathscr{F}_n - \mathscr{F}_n F'') \\ &= \Big(F'' - \sum_{n\in A} Z_n\mathscr{F}_n F''\Big) + \sum_{n\in A} Z_nQ_n\mathscr{F}_n \in [\mathbf{P}^p(E) \to \mathbf{P}^1(E)] \qquad (3.9) \end{aligned}$$

(here some $\mathscr{F}_n$ are preceded by the inclusion $\mathbf{P}^p(E) \to \mathbf{P}^1(E)$); F^0 is translation-invariant, by the preceding remarks and (2.1). Again by (2.1),

$$\begin{aligned} (F^0 - F'')Z_m &= \sum_{n\in A} Z_n(Q_n\mathscr{F}_n Z_m - \mathscr{F}_n F'' Z_m) = \sum_{n\in A} Z_n(\delta_{mn}Q_n - \mathscr{F}_n Z_m Q_m) \\ &= \sum_{n\in A} \delta_{mn} Z_n(Q_n - Q_m) = 0 \qquad \text{for all } m \in \mathbb{Z}. \end{aligned}$$

Therefore $F^0f = F''f = Ff$ for every trigonometric polynomial $f \in \mathbf{Tr}(E)$. Since $F \in [\mathbf{PC}(E) \to \mathbf{P}^1(E)]$, $F^0 \in [\mathbf{P}^p(E) \to \mathbf{P}^1(E)]$, and $\mathbf{Tr}(E)$ is dense in $\mathbf{PC}(E)$, we conclude that $F^0f = Ff$ for all $f \in \mathbf{PC}(E)$. Thus F_{p1} exists: namely, $F_{p1} = F^0$.

For every $f \in \mathbf{P}^p(E)$, (3.9) yields

$$F_{p1}f = F^0f = F'f + \sum_{n \in A} Z_n(Q_n \mathscr{F}_n f - \mathscr{F}_n F'f) \in \mathbf{P}^q(E).$$

By [1, Lemma 3.2, last paragraph with $p_1 = p_2 = p$, $q_1 = 1$, $q_2 = q$], we conclude that F_{pq} exists.

Remark. The theorem does not assert that $F_{pq} = F'$, i.e., that F' itself is bounded.

4. P-Memories and Related Mappings

We recall [1, p. 435] that a translation-invariant linear operator $F: \mathbf{PC}(E) \to \mathbf{P}^1(E)$ is a P-*memory* if it satisfies the following condition: for every interval $[a, b] \subset R$ and every $f \in \mathbf{PC}(E)$ that agrees with 0 on $[a - 1, b]$, Ff agrees with 0 on $[a, b]$. We now show that every P-memory is bounded; in fact, we prove a more general result.

Theorem 4.1. *Let* $\mathbf{X}$ *be one of the spaces* $\mathbf{P}^p(E)$, $1 \leqslant p < \infty$, *or* $\mathbf{PC}(E)$, *and let* q, $1 \leqslant q \leqslant \infty$, *be given. Suppose that the translation-invariant linear operator* $F': \mathbf{X} \to \mathbf{P}^q(E)$ *satisfies the condition: for every interval* $[a, b] \subset R$ *and every* $f \in \mathbf{X}$ *that agrees with* 0 *on* $[a - 1, b]$, $F'f$ *agrees with* 0 *on* $[a, b]$. *Then* F' *is bounded; and* $F' = F_{pq}$ *or* $F' = F_{Cq}$, *respectively, for some bounded* P-*memory* F. *In particular, every* P-*memory is bounded.*

Proof. (1) As in the proof of Theorem 3.4, an appeal to [1, Lemma 3.2] shows that there is no loss of generality in assuming, as we now shall, that $q = 1$.

There exists a function $\psi \in \mathbf{PC}(C)$ that agrees with 0 on $[-1, \frac{1}{6}]$ and satisfies

$$\psi + T_{-2/3}\psi + T_{-4/3}\psi = Z_0 1 \qquad \text{(the constant with value 1);} \qquad (4.1)$$

for instance, require $\psi(t) = 0$ for $t \in [-1, \frac{1}{6}]$, $\psi(t) = 6t - 1$ for $t \in [\frac{1}{6}, \frac{1}{3}]$, $\psi(t) = 1$ for $t \in [\frac{1}{3}, \frac{5}{6}]$, $\psi(t) = 6(1 - t)$ for $t \in [\frac{5}{6}, 1]$. Define the operator $\Psi \in [\mathbf{X} \to \mathbf{X}]$ by $\Psi f = \psi \cdot f$ for each $f \in \mathbf{X}$. Then Ψf agrees with 0 on $[-1, \frac{1}{6}]$, and (4.1) implies

$$\Psi + T_{-2/3}\Psi T_{2/3} + T_{-4/3}\Psi T_{4/3} = 1_{\mathbf{X}}. \qquad (4.2)$$

(2) Consider the set $A_{F'}$; it is finite, by Corollary 3.3. By the same corollary, it is enough to show that $A_{F'}$ is empty in order to conclude that F' is bounded. The remainder of the conclusion of the present theorem will then follow from the definition of P-memory and from [1, Lemma 3.2]. Assume then that $A_{F'} \neq \varnothing$, and choose a fixed $m \in A_{F'}$.

The restriction to $[0, \frac{1}{6}]$ of the finite family $(Z_n 1 : n \in A_{F'})$ is linearly independent. There exists therefore $\varphi \in \mathbf{Tr}(C)$—indeed, φ is a linear combination of the family itself—such that

$$\int_0^{1/6} \bar{\varphi} \cdot (Z_n 1) = \delta_{mn}, \qquad n \in A_{F'} \tag{4.3}$$

(the bar indicates value-wise conjugation). Define the operator $\Phi \in [\mathbf{P}^1(E) \to E]$ by $\Phi f = \int_0^{1/6} \bar{\varphi} \cdot f$ for each $f \in \mathbf{P}^1(E)$; (4.3) then implies

$$\Phi Z_n = \delta_{mn} 1_E, \qquad n \in A_{F'}. \tag{4.4}$$

(3) Since Ψf agrees with 0 on $[-1, \frac{1}{6}]$ for each $f \in \mathbf{X}$, the assumption on F' implies that $F'\Psi f$ agrees with 0 on $[0, \frac{1}{6}]$ for all such f, so that

$$\Phi F' \Psi = 0. \tag{4.5}$$

From (4.5), (4.4), and Corollary 3.3 we have

$$\begin{aligned} \mathscr{F}_m F' \Psi &= -\Phi F' \Psi + \sum_{n \in A_{F'}} \Phi Z_n \mathscr{F}_n F' \Psi \\ &= -\Phi \Big(F' - \sum_{n \in A_{F'}} Z_n \mathscr{F}_n F' \Big) \Psi \in [\mathbf{X} \to E]. \end{aligned}$$

By (4.2), (2.1), and the translation-invariance of F',

$$\begin{aligned} \mathscr{F}_m F' &= \mathscr{F}_m F' \Psi + \mathscr{F}_m F' T_{-2/3} \Psi T_{2/3} + \mathscr{F}_m F' T_{-4/3} \Psi T_{4/3} \\ &= \mathscr{F}_m F' \Psi (T_0 + e^{2\pi i m/3} T_{2/3} + e^{4\pi i m/3} T_{4/3}) \in [\mathbf{X} \to E], \end{aligned}$$

and this contradicts the assumption that $m \in A_{F'}$.

Corollary 4.2. *Suppose F is a* P*-memory, and that p, q, with $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, are given. If there exists a translation-invariant linear operator $F' : \mathbf{P}^p(E) \to \mathbf{P}^q(E)$ such that $F'f = Ff$ for all $f \in \mathbf{PC}(E)$, then (F is bounded and) F_{pq} exists.*

Proof. Theorems 4.1 and 3.4.

With a view to applications involving derivatives of periodic functions, we prove a refinement of Theorem 4.1, in which the operator is originally defined only for functions with mean value 0, i.e., for elements of $\mathbf{Y}_{\{0\}}$ only.

THEOREM 4.3. *Let X be one of the spaces $\mathbf{P}^p(E)$, $1 \leqslant p < \infty$, or $\mathbf{PC}(E)$, and let q, $1 \leqslant q \leqslant \infty$, be given. Set $\mathbf{X}_0 = \mathbf{X} \cap \mathbf{Y}_{\{0\}}$, considered as a (translation-invariant) closed subspace of $\mathbf{X}$. Suppose that the translation-invariant linear operator $F_0' \colon \mathbf{X}_0 \to \mathbf{P}^q(E)$ satisfies the condition that, for every interval $[a, b] \subset R$ and every $f \in \mathbf{X}_0$ that agrees with 0 on $[a-1, b]$, $F_0'f$ agrees with 0 on $[a, b]$. Then F_0' is bounded; and there exists a* P*-memory F such that F_{pq} or F_{Cq}, respectively, exists and satisfies $F_0'f = F_{pq}f$ or $F_0'f = F_{Cq}f$, respectively, for each $f \in \mathbf{X}_0$.*

Proof. We proceed by constructing a suitable extension $F' \colon \mathbf{X} \to \mathbf{P}^q(E)$ of F_0'.

Let $f \in \mathbf{X}$ be given. Suppose $[a, b]$, $[a', b'] \subset R$ are intervals with $[a', b'] \subset [a, b]$ and $b - a < 1$; in view of this inequality, there exist $f_0, f_0' \in \mathbf{X}_0$ such that f_0 agrees with f on $[a-1, b]$ and f_0' agrees with f on $[a'-1, b']$, and for any arbitrary choices of f_0, f_0' with these properties, f_0 and f_0' agree on $[a'-1, b']$, so that, by the assumption on F_0', $F_0'f_0$ and $F_0'f_0'$ agree on $[a', b']$. It is easy to infer that there exists a unique $g \in \mathbf{L}(E)$ such that for each interval $[a, b] \subset R$ with $b - a < 1$ and each $f_0 \in \mathbf{X}_0$ that agrees with f on $[a-1, b]$ the functions g and $F_0'f_0$ agree on $[a, b]$; it follows that $g \in \mathbf{P}^q(E)$.

We have thus constructed a mapping $F' \colon f \mapsto g \colon \mathbf{X} \to \mathbf{P}^q(E)$; the construction shows that F' is linear and translation-invariant. If $[a, b] \subset R$ is an interval and $f \in \mathbf{X}$ agrees with 0 on $[a-1, b]$, then $F'f$ agrees with $F_0'0 = 0$ on $[a, b]$, by the definition of F' if $b - a < 1$, and trivially if $b - a \geqslant 1$, since this implies $f = 0$. Thus, F' satisfies the assumptions of Theorem 4.1. If $f \in \mathbf{X}_0$, the construction shows that $F'f$ and $F_0'f$ agree on each short interval; hence, $F'f = F_0'f$. An application of Theorem 4.1 to F' then yields the conclusion.

Remark 1. The operator $F' \colon \mathbf{X} \to \mathbf{P}^q(E)$ that extends F_0' and satisfies the assumptions of Theorem 4.1 is obviously unique; consequently, the P-memory F in the statement of the theorem is also unique.

Remark 2. The conclusion of the theorem remains valid, and the proof only requires technical modifications, if $\mathbf{X}_0 = \mathbf{X} \cap \mathbf{Y}_{\{0\}}$ is replaced by $\mathbf{X} \cap \mathbf{Y}_K$ for any given *finite* set $K \subset \mathbb{Z}$.

5. An Example

EXAMPLE 5.1. (1) In this example, we assume $E = C$, and we shall write $\mathbf{P}^p$, $\mathbf{PC}$, without specifying the argument C.

We shall construct a (bounded) P-memory F. By [1, Lemma 3.7], F_{22} then exists, merely because $E = C$ is isomorphic to a Hilbert space and

$F \in [\mathbf{PC} \to \mathbf{P}^1]$ is translation-invariant. By [1, Lemmas 3.2 and 3.5], F_{pq} exists for all p, q with $1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty$, and $F_{\mathbf{C}q}$ exists for all q with $1 \leqslant q \leqslant 2$. In our example, F_{pq} or $F_{\mathbf{C}q}$ will not exist for any other values of p, q, nor will $F_{p\mathbf{C}}$ or $F_{\mathbf{CC}}$ exist for any values at all of p. Again by [1, Lemma 3.2], it will be enough to show that F_{p1} does not exist for any p, $1 \leqslant p < 2$, and that $F_{\mathbf{C}q}$ does not exist for any q, $2 < q \leqslant \infty$.

(2) There exists a function $u \in \mathbf{PC}$ that is periodic with period $\frac{2}{3}$ and has mean value 0, so that

$$u = \sum_{n \in \mathbb{Z}\setminus\{0\}} Z_{3n} \mathscr{F}_{3n} u$$

(the sum in $\mathbf{P}^2$), and a function $\lambda\colon \mathbb{Z}\setminus\{0\} \to \{-1, 1\}$ such that the function

$$v = \sum_{n \in \mathbb{Z}\setminus\{0\}} \lambda(n)\, Z_{3n} \mathscr{F}_{3n} u \tag{5.1}$$

(the sum in $\mathbf{P}^2$) satisfies $v \in \mathbf{P}^2 \setminus \bigcup_{q>2} \mathbf{P}^q$. The proof of this statement is in [2, Theorem (2.8)]; cf. [3, 14.3.3, 14.3.4]. The requirement that the mean value be 0 is a trivially permissible refinement. We let u, v, λ satisfying these conditions be fixed in what follows.

(3) Define $g \in \mathbf{P}^2$ by

$$g = -(2\pi i)^{-1} \sum_{n \in \mathbb{Z}\setminus\{0\}} n^{-1} \lambda(-n)\, Z_{3n} 1 \tag{5.2}$$

(the sum in $\mathbf{P}^2$); thus, g is also periodic with period $\frac{2}{3}$. Using (5.2) and (2.1) we find

$$\begin{aligned} \tfrac{1}{2} \int g \cdot (T_{-a} u) &= \sum_{n \in \mathbb{Z}\setminus\{0\}} (\mathscr{F}_{-3n} g)(\mathscr{F}_{3n} T_{-a} u) \\ &= (2\pi i)^{-1} \sum_{n \in \mathbb{Z}\setminus\{0\}} n^{-1} \lambda(n)\, e^{3\pi i n a} \mathscr{F}_{3n} u, \qquad a \in R. \end{aligned}$$

This computation, combined with (5.1), yields

$$\begin{aligned} \int g \cdot (T_{-b} - T_{-a}) u &= \sum_{n \in \mathbb{Z}\setminus\{0\}} (\pi i n)^{-1} \lambda(n) (e^{3\pi i n b} - e^{3\pi i n a})\, \mathscr{F}_{3n} u \\ &= 3 \int_a^b \sum_{n \in \mathbb{Z}\setminus\{0\}} \lambda(n)\, Z_{3n} \mathscr{F}_{3n} u \\ &= 3 \int_a^b v \qquad \text{for every interval } [a, b] \subset R. \end{aligned} \tag{5.3}$$

(4) Choose a continuously differentiable function $\varphi \in \mathbf{PC}$ such that φ agrees with 0 on [0, 1] and satisfies

$$\varphi + T_{2/3}\varphi + T_{4/3}\varphi = Z_0 1 \qquad \text{(the constant with value 1)}; \tag{5.4}$$

for instance, require $\varphi(t) = \cos^2(\frac{3}{2}\pi t)$ for $t \in [-1, -\frac{2}{3}]$, $\varphi(t) = 1$ for $t \in [-\frac{2}{3}, -\frac{1}{3}]$, $\varphi(t) = \sin^2(\frac{3}{2}\pi t)$ for $t \in [-\frac{1}{3}, 0]$, and $\varphi(t) = 0$ for $t \in [0, 1]$. Since φ is continuously differentiable—this assumption is in fact unnecessarily strong—we have

$$\sum_{k \in \mathbb{Z}} k^2 \mid \mathscr{F}_k \varphi \mid^2 < \infty. \tag{5.5}$$

It follows from (5.2) and (5.5) that

$$\mathscr{F}_n(\varphi \cdot g) = O(\mid n \mid^{-1}). \tag{5.6}$$

Indeed, for each $n \in \mathbb{Z} \backslash \{0\}$,

$$\begin{aligned}
&\mid \mathscr{F}_n(\varphi \cdot g) \mid \\
&= \Big| \sum_{k \in \mathbb{Z}} (\mathscr{F}_k \varphi)(\mathscr{F}_{n-k} g) \Big| \leqslant (3/2\pi) \sum_{k \in \mathbb{Z} \backslash \{n\}} \mid n - k \mid^{-1} \mid \mathscr{F}_k \varphi \mid \\
&= (3/2\pi) \Big(\mid n \mid^{-1} \mid \mathscr{F}_0 \varphi \mid + \sum_{k \in \mathbb{Z} \backslash \{0, n\}} \mid k \mid^{-1} \mid n - k \mid^{-1} \mid k \mid \mid \mathscr{F}_k \varphi \mid \Big) \\
&\leqslant (3/2\pi) \mid n \mid^{-1} \Big(\mid \mathscr{F}_0 \varphi \mid + \sum_{k \in \mathbb{Z} \backslash \{0, n\}} (\mid k \mid^{-1} + \mid n - k \mid^{-1}) \mid k \mid \mid \mathscr{F}_k \varphi \mid \Big) \\
&\leqslant (3/2\pi) \mid n \mid^{-1} \Big(\mid \mathscr{F}_0 \varphi \mid + 2 \Big(\sum_{k \in \mathbb{Z} \backslash \{0\}} k^{-2} \Big)^{1/2} \Big(\sum_{k \in \mathbb{Z} \backslash \{0\}} k^2 \mid \mathscr{F}_k \varphi \mid^2 \Big)^{1/2} \Big).
\end{aligned}$$

(5) Now the function $\varphi \cdot g \in \mathbf{P}^2$ agrees with 0 on]0, 1[and satisfies (5.6). By [1, Corollary 7.2] there exists a (bounded) P-memory F with $FZ_0 = 0$ and such that $G_F = \varphi \cdot g$; by the same corollary, this last equality means, in particular, that

$$\int_a^b Fu = \int \varphi \cdot g \cdot (T_{-b} - T_{-a}) u \qquad \text{for every interval } [a, b] \subset R. \tag{5.7}$$

On the other hand, g and u are periodic with period $\frac{2}{3}$. Therefore, (5.4) implies

$$\begin{aligned}
\int g \cdot (T_{-b} - T_{-a}) u &= \int (\varphi + T_{2/3}\varphi + T_{4/3}\varphi) \cdot g \cdot (T_{-b} - T_{-a}) u \\
&= \int (T_0 + T_{2/3} + T_{4/3})(\varphi \cdot g \cdot (T_{-b} - T_{-a}) u) \\
&= 3 \int \varphi \cdot g \cdot (T_{-b} - T_{-a}) u
\end{aligned}$$

$$\text{for every interval } [a, b] \subset R. \tag{5.8}$$

Combining (5.7), (5.8), (5.3), we find $\int_a^b Fu = \frac{1}{3}\int g \cdot (T_{-b} - T_{-a})\, u = \int_a^b v$ for all $[a, b] \subset R$, so that $Fu = v$. Since $u \in \mathbf{PC}$ and $v \in \mathbf{P}^2 \backslash \bigcup_{q>2} \mathbf{P}^q$, we conclude that $F_{\mathbf{C}q}$ does not exist for any q, $2 < q \leqslant \infty$; a fortiori, $F_{\mathbf{CC}}$ does not exist.

(6) By [1, Corollary 7.3], F^*, as defined in [1, p. 434], exists and is a bounded P-memory with $F^*Z_0 = 0$, and $G_{F^*} = \bar{\varphi} \cdot \bar{g}$ (the bars again indicate valuewise conjugation). By [1, Corollary 7.2] and (5.7), (5.8), (5.3), we find

$$\begin{aligned}\int_a^b F^*\bar{u} &= \int \bar{\varphi} \cdot \bar{g} \cdot (T_{-b} - T_{-a})\bar{u} \\ &= \tfrac{1}{3} \int \bar{g} \cdot (T_{-b} - T_{-a})\bar{u} = \int_a^b \bar{v} \qquad \text{for every } [a, b] \subset R,\end{aligned}$$

so that $F^*\bar{u} = \bar{v}$. Arguing as in the preceding paragraph, we conclude that $F^*_{\mathbf{C}p'}$ does not exist for any p', $2 < p' \leqslant \infty$. By [1, Lemma 3.6], we find that F_{p1} does not exist for any p, $1 \leqslant p < 2$. This completes the proof of the properties claimed for F.

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